

GENERIC POINTS IN SYSTEMS OF SPECIFICATION AND BANACH VALUED BIRKHOFF ERGODIC AVERAGE

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ABSTRACT. We prove that systems satisfying the specification property are saturated in the sense that the topological entropy of the set of generic points of any invariant measure is equal to the measure-theoretic entropy of the measure. We study Banach valued Birkhoff ergodic averages and obtain a variational principle for its topological entropy spectrum. As application, we examine a particular example concerning with the set of real numbers for which the frequencies of occurrences in their dyadic expansions of infinitely many words are prescribed. This relies on our explicit determination of a maximal entropy measure.

1. INTRODUCTION

By dynamical system (X, T) , we mean a continuous transformation $T : X \rightarrow X$ on a compact metric space X with metric d . We shall adopt the notion of topological entropy introduced by Bowen ([9], recalled in the section 2), denoted by h_{top} , to describe the sizes of sets in X . We denote by \mathcal{M}_{inv} the set of all T -invariant probability Borel measures on X and by \mathcal{M}_{erg} its subset of all ergodic measures. The measure-theoretic entropy of μ in \mathcal{M}_{inv} is denoted by h_μ .

Let us first recall some notions like generic points, saturated property and the specification property which are quite known in dynamical systems nowadays.

For $\mu \in \mathcal{M}_{\text{inv}}$, the set G_μ of μ -generic points is defined by

$$G_\mu := \left\{ x \in X : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j x} \xrightarrow{w^*} \mu \right\},$$

where $\xrightarrow{w^*}$ stands for the weak star convergence of the measures.

A dynamical system (X, T) is said to be *saturated* if for any $\mu \in \mathcal{M}_{\text{inv}}$, we have $h_{\text{top}}(G_\mu) = h_\mu$.

Bowen ([8]) proved that on any dynamical system, we have $h_{\text{top}}(G_\mu) \leq h_\mu$ for any $\mu \in \mathcal{M}_{\text{inv}}$. So, saturatedness means that G_μ is of optimal topological entropy. One of our main results is to prove that systems of specification share this saturatedness.

A dynamical system (X, T) is said to satisfy the *specification property* if for any $\epsilon > 0$ there exists an integer $m(\epsilon) \geq 1$ having the property that for any integer $k \geq 2$, for any k points x_1, \dots, x_k in X , and for any integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$$

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with $a_i - b_{i-1} \geq m(\epsilon)$ ($\forall 2 \leq i \leq k$), there exists a point $y \in X$ such that

$$d(T^{a_i+n}y, T^n x_i) < \epsilon \quad (\forall 0 \leq n \leq b_i - a_i, \quad \forall 1 \leq i \leq k).$$

The specification property was introduced by Bowen ([8]) who required that y is periodic. But the present day tradition doesn't require this. The specification property implies the topological mixing. Blokh ([7]) proved that these two properties are equivalent for continuous interval transformations. Mixing subshifts of finite type satisfy the specification property. In general, a subshift satisfies the specification if for any admissible words u and v there exists a word w with $|w| \leq k$ (some constant k) such that uwv is admissible. For β -shifts defined by $T_\beta x = \beta x \pmod{1}$, there is only a countable number of β 's such that the β shifts admit Markov partition (i.e. subshifts of finite type), but an uncountable number of β 's such that the β -shifts satisfy the specification property ([27]).

Our first result is stated as follows.

Theorem 1.1. *If the dynamical system (X, T) satisfies the specification property, then it is saturated.*

As application, we study Banach-valued Birkhoff averages for saturated systems. Let \mathbb{B} be a real Banach space and \mathbb{B}^* its dual space, their duality being denoted by $\langle \cdot, \cdot \rangle$. We consider \mathbb{B}^* as a locally convex topological space with the weak star topology $\sigma(\mathbb{B}^*, \mathbb{B})$. For any \mathbb{B}^* -valued continuous function $\Phi : X \rightarrow \mathbb{B}^*$, we consider its Birkhoff ergodic averages

$$A_n \Phi(x) = \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x) \quad (n \geq 1).$$

We would like to know the asymptotic behavior of $A_n \Phi(x)$ in the $\sigma(\mathbb{B}^*, \mathbb{B})$ -topology for different points $x \in X$.

Let us state the problem we are studying as follows. Fix a subset $E \subset \mathbb{B}$. For a sequence $\{\xi_n\} \subset \mathbb{B}^*$ and a point $\xi \in \mathbb{B}^*$, we denote by $\limsup_{n \rightarrow \infty} \overset{E}{\xi_n} \leq \xi$ the fact

$$\limsup_{n \rightarrow \infty} \langle \xi_n, w \rangle \leq \langle \xi, w \rangle \quad \text{for all } w \in E.$$

The meaning of " $\overset{E}{\limsup}$ " is obvious. It is clear that $\limsup_{n \rightarrow \infty} \overset{\mathbb{B}}{\xi_n} \leq \xi$, or equivalently $\limsup_{n \rightarrow \infty} \overset{\mathbb{B}}{\xi_n} = \xi$, means ξ_n converges to ξ in the weak star topology $\sigma(\mathbb{B}^*, \mathbb{B})$.

Let $\alpha \in \mathbb{B}^*$ and $E \subset \mathbb{B}$. The object of our study is the set

$$X_\Phi(\alpha; E) = \left\{ x \in X : \limsup_{n \rightarrow \infty} A_n \Phi(x) \overset{E}{\leq} \alpha \right\}.$$

The set $X_\Phi(\alpha; \mathbb{B})$ will be simply denoted by $X_\Phi(\alpha)$. This is the set of points $x \in X$ such that $\lim_{n \rightarrow \infty} A_n \Phi(x) = \alpha$ in $\sigma(\mathbb{B}^*, \mathbb{B})$ -topology. If \widehat{E} denotes the convex cone of E which consists of all $aw' + bw''$ with $a \geq 0, b \geq 0$ and $w' \in E, w'' \in E$, then $X_\Phi(\alpha, E) = X_\Phi(\alpha, \widehat{E})$. So we may always assume that E is a convex cone. If E is symmetric in the sense that $E = -E$, then we have

$$X_\Phi(\alpha, E) = \left\{ x \in X : \lim_{n \rightarrow \infty} A_n \Phi(x) \overset{E}{=} \alpha \right\}.$$

By entropy spectrum we mean the function

$$\mathcal{E}_\Phi^E(\alpha) := h_{\text{top}}(X_\Phi(\alpha; E)).$$

Invariant measures will be involved in the study of the entropy spectrum $\mathcal{E}_\Phi^E(\alpha)$. We set

$$\mathcal{M}_\Phi(\alpha; E) = \left\{ \mu \in \mathcal{M}_{\text{inv}} : \int \Phi d\mu \stackrel{E}{\leq} \alpha \right\}$$

where $\int \Phi d\mu$ denotes the vector-valued integral in Pettis' sense (see [26]) and the inequality " $\stackrel{E}{\leq}$ " means

$$\int \langle \Phi, w \rangle d\mu \leq \langle \alpha, w \rangle \quad \text{for all } w \in E.$$

For saturated systems, we prove the following variational principle.

Theorem 1.2. *Suppose that the dynamical system (X, T) is saturated. Then*

- (a) *If $\mathcal{M}_\Phi(\alpha; E) = \emptyset$, we have $X_\Phi(\alpha, E) = \emptyset$.*
- (b) *If $\mathcal{M}_\Phi(\alpha; E) \neq \emptyset$, we have*

$$h_{\text{top}}(X_\Phi(\alpha; E)) = \sup_{\mu \in \mathcal{M}_\Phi(\alpha; E)} h_\mu. \quad (1.1)$$

When \mathbb{B} is a finite dimensional Euclidean space \mathbb{R}^d and $E = \mathbb{R}^d$, the variational principle (1.1) with $E = \mathbb{R}^d$ was proved in [15, 16] for subshifts of finite type, then for conformal repellers ([17]) and later generalized to systems with specification property [28]. There are other works assuming that Φ is regular (Hölder for example). See [6, 14] for classical discussions, [2, 3, 20, 21, 23, 29] for recent developments on Birkhoff averages, and [1, 5, 10, 12, 4] for the multifractal analysis of measures.

The study of infinite dimensional Birkhoff averages is a new subject. We point out that [24] provides another point of view, i.e. the thermodynamical point of view which was first introduced by physicists.

The above variational principle (1.1), when $E = \mathbb{B}$, is easy to generalize to the following setting. Let Ψ be a continuous function defined on the closed convex hull of the image $\Phi(X)$ of Φ into a topological space Y . For given Φ , Ψ , and $\beta \in Y$, we set

$$X_\Phi^\Psi(\beta) = \left\{ x \in X : \lim_{n \rightarrow \infty} \Psi(A_n \Phi(x)) = \beta \right\}$$

and

$$\mathcal{M}_\Phi^\Psi(\beta) = \left\{ \mu \in \mathcal{M}_{\text{inv}} : \Psi \left(\int \Phi d\mu \right) = \beta \right\}.$$

We also set

$$\widehat{X}_\Phi^\Psi(\beta) = \left\{ x \in X : \Psi \left(\lim_{n \rightarrow \infty} A_n \Phi(x) \right) = \beta \right\} = \bigcup_{\alpha: \Psi(\alpha) = \beta} X_\Phi(\alpha).$$

It is clear that $\widehat{X}_\Phi^\Psi(\beta)$ is a subset of $X_\Phi^\Psi(\beta)$.

Theorem 1.3. *Suppose that the dynamical system (X, T) is saturated. Then*

- (1) *if $\mathcal{M}_\Phi^\Psi(\beta) = \emptyset$, we have $X_\Phi^\Psi(\beta) = \emptyset$,*
- (2) *if $\mathcal{M}_\Phi^\Psi(\beta) \neq \emptyset$, we have*

$$h_{\text{top}}(X_\Phi^\Psi(\beta)) = h_{\text{top}}(\widehat{X}_\Phi^\Psi(\beta)) = \sup_{\mu \in \mathcal{M}_\Phi^\Psi(\beta)} h_\mu. \quad (1.2)$$

This generalized variational principle (1.2) allows us to study generalized ergodic limits like

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=0}^{n-1} \Phi(T^j x)}{\sum_{j=0}^{n-1} g(T^j x)}, \quad (1.3)$$

where $g : X \rightarrow \mathbb{R}^+$ is a continuous positive function. It suffices to apply (1.2) to Φ replaced by (Φ, g) and Ψ defined by $\Psi(x, y) = x/y$, with $x \in \mathbb{B}^*$ and $y \in \mathbb{R}^+$.

It also allows us to study the set of points $x \in X$ for which the limits $A_\infty \Phi(x) = \lim_{n \rightarrow \infty} A_n \Phi(x)$ verify the equation

$$\Psi(A_\infty \Phi(x)) = \beta. \quad (1.4)$$

The present infinite dimensional version of variational principle would have many interesting applications. We will just illustrate the usefulness of the variational principle by the following study of frequencies of blocks in the dyadic development of real numbers. It can be reviewed as an infinitely multi-recurrence problem.

Let us state the question to which we can answer. All but a countable number of real numbers $t \in [0, 1]$ can be uniquely developed as follows

$$t = \sum_{n=1}^{\infty} \frac{t_n}{2^n} \quad (t_n \in \{0, 1\}).$$

Let $k \geq 1$. We write 0^k for the block of k consecutive zeroes and we define the 0^n -frequency of t as the limit (if it exists)

$$f(t, k) = \lim_{n \rightarrow \infty} \frac{1}{n} \# \{1 \leq j \leq n : t_j t_{j+1} \cdots t_{j+k-1} = 0^k\}.$$

Let (a_1, a_2, \dots) be a sequence of non-negative numbers. We denote by $S(a_1, a_2, \dots)$ the set of all numbers $t \in [0, 1]$ such that $f(t, k) = a_k$ for all $k \geq 1$. As a consequence of the variational principle (1.1), we prove

Theorem 1.4. *The set $S(a_1, a_2, \dots)$ is non-empty if and only if the following condition is satisfied*

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \quad (1.5)$$

If the condition (1.5) is fulfilled, we have

$$h_{\text{top}}(S(a_1, a_2, \dots)) = -h(1 - a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2}) \quad (1.6)$$

where $h(x) = -x \log x$.

Furthermore, it is proved that there is a unique maximal entropy measure, which is completely determined (see Lemma 5.6).

The paper is organized as follows. In the section 2, we give some preliminaries. In the section 3, we prove Theorem 1.1. In the section 4, we prove the theorems 1.2 and 1.3 and examine the case where $\mathbb{B} = \ell^1(\mathbb{Z})$. In the section 5, we apply the variational principle (1.1) to the study of the recurrence into an infinite number of cylinders of the symbolic dynamics. Especially, we study the set of orbits whose recurrences into infinitely many cylinders are prescribed. This relies on the explicit determination of a maximal entropy measure which, by definition, maximizes the supremum in (1.1).

2. PRELIMINARY

Before proving the main theorems, we wish to recall the notions of topological entropy, the Bowen lemma and two propositions about the measure-theoretic entropy.

Recall that X is a compact metric space with its metric d and that $T : X \rightarrow X$ is a continuous transformation on X . For any integer $n \geq 1$ we define the Bowen metric d_n on X by

$$d_n(x, y) = \max_{0 \leq j < n} d(T^j x, T^j y).$$

For any $\epsilon > 0$, we will denote by $B_n(x, \epsilon)$ the open d_n -ball centered at x of radius ϵ .

2.1. Topological entropy and Bowen lemma.

Let $Z \subset X$ be a subset of X . Let $\epsilon > 0$. We say that a collection (at most countable) $R = \{B_{n_i}(x_i, \epsilon)\}$ covers Z if $Z \subset \bigcup_i B_{n_i}(x_i, \epsilon)$. For such a collection R , we put $n(R) = \min_i n_i$. Let $s \geq 0$. Define

$$H_n^s(Z, \epsilon) = \inf_R \sum_i \exp(-sn_i),$$

where the infimum is taken over all covers R of Z with $n(R) \geq n$. The quantity $H_n^s(Z, \epsilon)$ is non-decreasing as a function of n , so the following limit exists

$$H^s(Z, \epsilon) = \lim_{n \rightarrow \infty} H_n^s(Z, \epsilon).$$

For the quantity $H^s(Z, \epsilon)$ considered as a function of s , there exists a critical value, which we denote by $h_{\text{top}}(Z, \epsilon)$, such that

$$H^s(Z, \epsilon) = \begin{cases} +\infty, & s < h_{\text{top}}(Z, \epsilon) \\ 0, & s > h_{\text{top}}(Z, \epsilon). \end{cases}$$

One can prove that the following limit exists

$$h_{\text{top}}(Z) = \lim_{\epsilon \rightarrow 0} h_{\text{top}}(Z, \epsilon).$$

The quantity $h_{\text{top}}(Z)$ is called the *topological entropy* of Z ([9]).

For $x \in X$, we denote by $V(x)$ the set of all weak limits of the sequence of probability measures $n^{-1} \sum_{j=0}^{n-1} \delta_{T^j x}$. It is clear that $V(x) \neq \emptyset$ and $V(x) \subset \mathcal{M}_{\text{inv}}$ for any x . The following Bowen lemma is one of the key lemmas for proving the variational principle.

Lemma 2.1 (Bowen [8]). *For $t \geq 0$, consider the set*

$$B^{(t)} = \{x \in X : \exists \mu \in V(x) \text{ satisfying } h_\mu \leq t\}.$$

Then $h_{\text{top}}(B^{(t)}) \leq t$.

Let $\mu \in \mathcal{M}_{\text{inv}}$ be an invariant measure. A point $x \in X$ such that $V(x) = \{\mu\}$ is said to be generic for μ . Recall our definition of G_μ , we know that G_μ is the set of all generic points for μ . Bowen proved that $h_{\text{top}}(G_\mu) \leq h_\mu$ for any invariant measure. This assertion can be deduced by using Lemma 2.1. In fact, the reason is that $x \in G_\mu$ implies $\mu \in V(x)$. Bowen also proved that the inequality becomes equality when μ is ergodic. However, in general, we do not have the equality $h_{\text{top}}(G_\mu) = h_\mu$ (saturatedness) and it is even possible that $G_\mu = \emptyset$. Cajar [11] proved that full symbolic spaces are saturated. Concerning the μ -measure of G_μ , it is well known that $\mu(G_\mu) = 1$ or 0 according to whether μ is ergodic or not (see [13]).

2.2. Two propositions about the measure-theoretic entropy.

We denote by $C(X)$ the set of continuous functions on X , by $\mathcal{M} = \mathcal{M}(X)$ the set of all Borel probability measures.

In the sequel, we fix a sequence $(p_i)_{i \geq 1}$ such that $p_i > 0$ for all $i \geq 1$ and $\sum_{i=1}^{\infty} p_i = 1$ (for example, $p_i = 2^{-i}$ will do). Suppose that $s_n = (s_{n,i})_{i \geq 1}$ ($n = 1, 2, \dots$) is a sequence of elements in ℓ^∞ . It is obvious that s_n converges to $\alpha = (\alpha_i)_{i \geq 1} \in \ell^\infty$ in the weak star topology (i.e. each coordinate converges) is equivalent to

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} p_i |s_{n,i} - \alpha_i| = 0.$$

We also fix a sequence of continuous functions $\{\Phi_1, \Phi_2, \dots\}$ which is dense in the unit ball of $C(X)$. Write $\Phi = (\Phi_1, \Phi_2, \dots)$. It is evident that $\Phi : X \rightarrow \ell^\infty$ is continuous when ℓ^∞ is equipped with its weak star topology. Fix an invariant measure $\mu \in \mathcal{M}_{\text{inv}}$. Let

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_i, \dots) \quad \text{where} \quad \alpha_i = \int \Phi_i d\mu.$$

The set of generic points G_μ can be described as follows

$$G_\mu = \left\{ x \in X : \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} p_i |A_n \Phi_i - \alpha_i| = 0 \right\} = X_\Phi(\alpha). \quad (2.1)$$

It is well known that the weak topology of \mathcal{M} is compatible with the topology induced by the metric

$$\tilde{d}(\mu, \nu) = \sum_{i=1}^{\infty} p_i \left| \int \Phi_i d\mu - \int \Phi_i d\nu \right| \quad (2.2)$$

where both $(p_i)_{i \geq 1}$ and $\{\Phi_i\}_{i \geq 1}$ are chosen as above.

The following two results will be useful for us.

Proposition 2.2 (Young [31]). *For any $\mu \in \mathcal{M}_{\text{inv}}$ and any numbers $0 < \delta < 1$ and $0 < \theta < 1$, there exist an invariant measure ν which is a finite convex combination of ergodic measures, i.e.*

$$\nu = \sum_{k=1}^r \lambda_k \nu_k, \quad \text{where} \quad \lambda_k > 0, \quad \sum_{k=1}^r \lambda_k = 1, \quad \nu_k \in \mathcal{M}_{\text{erg}}, \quad r \in \mathbb{N}^+$$

such that

$$\tilde{d}(\mu, \nu) < \delta, \quad h_\nu \geq h_\mu - \theta.$$

This is a consequence of the following result due to Jacobs (see [30], p. 186). Let $\mu \in \mathcal{M}_{\text{inv}}$ be an invariant measure which has the ergodic decomposition $\mu = \int_{\mathcal{M}_{\text{erg}}} \tau d\pi(\tau)$ where π is a Borel probability measure on \mathcal{M}_{erg} . Then we have

$$h_\mu = \int_{\mathcal{M}_{\text{erg}}} h_\tau d\pi(\tau).$$

Proposition 2.3 (Katok [18]). *Let $\mu \in \mathcal{M}_{\text{erg}}$ be an ergodic invariant measure. For $\epsilon > 0$ and $\delta > 0$, let $r_n(\epsilon, \delta, \mu)$ denote the minimum number of ϵ -balls in the Bowen*

metric d_n whose union has μ -measure more than or equal to $1 - \delta$. Then for each $\delta > 0$ we have

$$h_\mu = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(\epsilon, \delta, \mu) = \lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log r_n(\epsilon, \delta, \mu).$$

In [18], it was assumed that $T : X \rightarrow X$ is a homeomorphism. But the proof in [18] works for the transformations we are studying.

3. SYSTEMS WITH SPECIFICATION PROPERTY ARE SATURATED

In this section, we prove Theorem 1.1 which says that every system satisfying the specification property is saturated. Because of Bowen's lemma (Lemma 2.1), we have only to show $h_{\text{top}}(G_\mu) \geq h_\mu$. The idea of the proof appeared in [15, 16] and was developed in [28]. It consists of constructing the so-called dynamical Moran sets which approximate the set of generic points G_μ .

3.1. Dynamical Moran sets and their entropies.

Fix $\epsilon > 0$. Let $\{m_k\}_{k \geq 1}$ be the sequence of integers defined by $m_k = m(2^{-k}\epsilon)$ which is the constant appeared in the definition of the specification property ($k = 1, 2, \dots$). Let $\{W_k\}_{k \geq 1}$ be a sequence of finite sets in X and $\{n_k\}_{k \geq 1}$ be a sequence of positive integers. Assume that

$$d_{n_k}(x, y) \geq 5\epsilon \quad (\forall x, y \in W_k \quad x \neq y). \quad (3.1)$$

Let $\{N_k\}_{k \geq 1}$ be another sequence of positive integers with $N_1 = 1$. Using these data, we are going to construct a compact set of Cantor type, called a dynamical Moran set, which will be denoted by $F = F(\epsilon, \{W_k\}, \{n_k\}, \{N_k\})$. We will give an estimate for its topological entropy.

Denote

$$M_k = \#W_k.$$

Fix $k \geq 1$. For any N_k points x_1, \dots, x_{N_k} in W_k i.e. $(x_1, \dots, x_{N_k}) \in W_k^{N_k}$, we choose a point $y(x_1, \dots, x_{N_k}) \in X$, which does exist by the specification property, such that

$$d_{n_k}(x_s, T^{a_s}y) < \frac{\epsilon}{2^k} \quad (s = 1, \dots, N_k) \quad (3.2)$$

where

$$a_s = (s-1)(n_k + m_k).$$

Both (3.1) and (3.2) imply that for two distinct points (x_1, \dots, x_{N_k}) and $(\bar{x}_1, \dots, \bar{x}_{N_k})$ in $W_k^{N_k}$ we have

$$d_{t_k}(y(x_1, \dots, x_{N_k}), y(\bar{x}_1, \dots, \bar{x}_{N_k})) > 4\epsilon \quad (3.3)$$

where $t_k = a_{N_k} + n_k$, i.e.

$$t_k = (N_k - 1)m_k + N_k n_k.$$

In fact, let $y = y(x_1, \dots, x_{N_k})$ and $\bar{y} = y(\bar{x}_1, \dots, \bar{x}_{N_k})$. Suppose $x_s \neq \bar{x}_s$ for some $s \in \{1, \dots, N_k\}$. Then

$$\begin{aligned} d_{t_k}(y, \bar{y}) &\geq d_{n_k}(T^{a_s}y, T^{a_s}\bar{y}) \\ &\geq d_{n_k}(x_s, \bar{x}_s) - d_{n_k}(x_s, T^{a_s}y) - d_{n_k}(\bar{x}_s, T^{a_s}\bar{y}) \\ &> 5\epsilon - \epsilon/2 - \epsilon/2 = 4\epsilon. \end{aligned}$$

Let

$$D_1 = W_1, \quad D_k = \left\{ y(x_1, \dots, x_{N_k}) : (x_1, \dots, x_{N_k}) \in W_k^{N_k} \right\} \quad (\forall k \geq 2).$$

Now define recursively L_k and ℓ_k as follows. Let

$$L_1 = D_1, \quad \ell_1 = n_1.$$

For any $x \in L_k$ and any $y \in D_{k+1}$ ($k \geq 1$), by the specification property, we can find a point $z(x, y) \in X$ such that

$$d_{\ell_k}(z(x, y), x) < \frac{\epsilon}{2^{k+1}}, \quad d_{t_{k+1}}(T^{\ell_k+m_{k+1}}z(x, y), y) < \frac{\epsilon}{2^{k+1}}.$$

We will choose one and only one such $z(x, y)$ and call it the descend from $x \in L_k$ through $y \in D_{k+1}$. Let

$$\begin{aligned} L_{k+1} &= \{z(x, y) : x \in L_k, y \in D_{k+1}\}, \\ \ell_{k+1} &= \ell_k + m_{k+1} + t_{k+1} = N_1 n_1 + \sum_{i=2}^{k+1} N_i(m_i + n_i). \end{aligned}$$

Observe that for any $x \in L_k$ and for all $y, \bar{y} \in D_{k+1}$ with $y \neq \bar{y}$, we have

$$d_{\ell_k}(z(x, y), z(x, \bar{y})) < \frac{\epsilon}{2^k} \quad (k \geq 1), \quad (3.4)$$

and for any $x, \bar{x} \in L_k$ and $y, \bar{y} \in D_{k+1}$ with $(x, y) \neq (\bar{x}, \bar{y})$, we have

$$d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) > 3\epsilon \quad (k \geq 1). \quad (3.5)$$

The fact (3.4) is obvious. To prove (3.5), first remark that $d_{\ell_1}(z, \bar{z}) \geq 5\epsilon > 4\epsilon$ for any $z, \bar{z} \in L_1$ with $z \neq \bar{z}$, and that for any $x, \bar{x} \in L_k$ and $y, \bar{y} \in D_{k+1}$ with $(x, y) \neq (\bar{x}, \bar{y})$ we have

$$d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) \geq d_{\ell_k}(x, \bar{x}) - d_{\ell_k}(z(x, y), x) - d_{\ell_k}(z(\bar{x}, \bar{y}), \bar{x})$$

and

$$\begin{aligned} & d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) \\ & \geq d_{t_{k+1}}(y, \bar{y}) - d_{t_{k+1}}(T^{\ell_k+m_{k+1}}z(x, y), y) - d_{t_{k+1}}(T^{\ell_k+m_{k+1}}z(\bar{x}, \bar{y}), \bar{y}). \end{aligned}$$

Now using the above two inequalities, we prove (3.5) by induction. For any $x, \bar{x} \in L_1$ and $y, \bar{y} \in D_2$ with either $x \neq \bar{x}$ or $y \neq \bar{y}$, we have

$$d_{\ell_2}(z(x, y), z(\bar{x}, \bar{y})) > 4\epsilon - \frac{\epsilon}{2^2} - \frac{\epsilon}{2^2} = 4\epsilon - \frac{\epsilon}{2}.$$

Suppose we have obtained that

$$d_{\ell_k}(z(x, y), z(\bar{x}, \bar{y})) > 4\epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2^2} - \cdots - \frac{\epsilon}{2^{k-1}}.$$

Then for any $x, \bar{x} \in L_k$ and $y, \bar{y} \in D_{k+1}$ with $(x, y) \neq (\bar{x}, \bar{y})$ we have

$$\begin{aligned} d_{\ell_{k+1}}(z(x, y), z(\bar{x}, \bar{y})) &> 4\epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2^2} - \cdots - \frac{\epsilon}{2^{k-1}} - \frac{\epsilon}{2^{k+1}} - \frac{\epsilon}{2^{k+1}} \\ &= 4\epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2^2} - \cdots - \frac{\epsilon}{2^{k-1}} - \frac{\epsilon}{2^k} > 3\epsilon \end{aligned}$$

Now define our dynamical Moran set

$$F = F(\epsilon, \{W_k\}, \{n_k\}, \{N_k\}) = \bigcap_{k=1}^{\infty} F_k,$$

where

$$F_k = \bigcup_{x \in L_k} \overline{B}_{\ell_k}(x, \epsilon 2^{-(k-1)})$$

($\overline{B}(x, r)$ denoting the closed ball of center x and radius r). The set F is Cantor-like because for any distinct points $x', x'' \in L_k$, by (3.5) we have

$$\overline{B}_{\ell_k}(x', \epsilon 2^{-(k-1)}) \cap \overline{B}_{\ell_k}(x'', \epsilon 2^{-(k-1)}) = \emptyset$$

and if $z \in L_{k+1}$ descends from $x \in L_k$, by (3.4) we have

$$\overline{B}_{\ell_{k+1}}(z, \epsilon 2^{-k}) \subseteq \overline{B}_{\ell_k}(x, \epsilon 2^{-(k-1)}).$$

Proposition 3.1 (Entropy of F). *For any integer $n \geq 1$, let $k = k(n) \geq 1$ and $0 \leq p = p(n) < N_{k+1}$ be the unique integers such that*

$$\ell_k + p(m_{k+1} + n_{k+1}) < n \leq \ell_k + (p+1)(m_{k+1} + n_{k+1}).$$

We have

$$h_{\text{top}}(F) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} (N_1 \log M_1 + \cdots + N_k \log M_k + p \log M_{k+1}).$$

Proof. For every $k \geq 1$, consider the discrete measure σ_k concentrated on F_k

$$\sigma_k = \frac{1}{\#L_k} \sum_{x \in L_k} \delta_x.$$

It can be proved that σ_k converges in the weak star topology to a probability measure σ concentrated on F . Moreover, for sufficiently large n and every point $x \in X$ such that $B_n(x, \epsilon/2) \cap F \neq \emptyset$, we have

$$\sigma(B_n(x, \epsilon/2)) \leq \frac{1}{\#(L_k)M_{k+1}^p} = \frac{1}{M_1^{N_1} \cdots M_k^{N_k} M_{k+1}^p}.$$

(see [28]). Then we apply the mass distribution principle to estimate the entropy. \square

3.2. Box-counting of G_μ .

Recall that $\alpha = (\alpha_i)_{i \geq 1} \in \ell^\infty$ and $\Phi = (\Phi_i)$ is a dense sequence in the unit ball of $C(X)$. For $\delta > 0$ and $n \geq 1$, define

$$X_\Phi(\alpha, \delta, n) = \left\{ x \in X : \sum_{i=1}^n p_i |A_n \Phi_i(x) - \alpha_i| < \delta \right\}.$$

For $\epsilon > 0$, let $N(\alpha, \delta, n, \epsilon)$ denote the minimal number of balls $B_n(x, \epsilon)$ to cover the set $X_\Phi(\alpha, \delta, n)$. Define

$$\Lambda_\Phi(\alpha) := \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha, \delta, n, \epsilon) \quad (3.6)$$

By the same argument in [15] (p. 884-885), we can prove the existence of the limits, and the following equality:

$$\Lambda_\Phi(\alpha) = \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha, \delta, n, \epsilon).$$

Proposition 3.2. $\Lambda_\Phi(\alpha) \geq h_\mu$.

Proof. It suffices to prove $\Lambda_\Phi(\alpha) \geq h_\mu - 4\theta$ for any $\theta > 0$. For each $i \geq 1$, define the variation of Φ_i by

$$\text{var}(\Phi_i, \epsilon) = \sup_{d(x, y) < \epsilon} |\Phi_i(x) - \Phi_i(y)|.$$

By the compactness of X and the continuity of Φ_i , $\lim_{\epsilon \rightarrow 0} \text{var}(\Phi_i, \epsilon) \rightarrow 0$. So

$$\lim_{\epsilon \rightarrow 0} \sum_{i=1}^{+\infty} p_i \text{var}(\Phi_i, \epsilon) \rightarrow 0.$$

This, together with (3.6), allows us to choose $\epsilon > 0$ and $\delta > 0$ such that

$$\sum_{i=1}^{+\infty} p_i \text{var}(\Phi_i, \epsilon) < \delta < \theta \quad (3.7)$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha, 5\delta, n, \epsilon) < \Lambda_\Phi(\alpha) + \theta. \quad (3.8)$$

For the measure μ , take an invariant measure $\nu = \sum_{k=1}^r \lambda_k \nu_k$ having the properties stated in Proposition 2.2. For $1 \leq k \leq r$ and $N \geq 1$, set

$$Y_k(N) = \left\{ x \in X : \sum_{i=1}^{\infty} p_i \left| A_n \Phi_i(x) - \int \Phi_i d\nu_k \right| < \delta \quad (\forall n \geq N) \right\}.$$

Since ν_k is ergodic, by the Birkhoff theorem, we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} p_i \left| A_n \Phi_i(x) - \int \Phi_i d\nu_k \right| = 0 \quad \nu_k\text{-a.e.} \quad (3.9)$$

Then by the Egorov theorem, there exists a set with ν_k -measure greater than $1 - \theta$ on which the above limit (3.9) is uniform. Therefore, if N is sufficiently large, we have

$$\nu_k(Y_k(N)) > 1 - \theta \quad (\forall k = 1, \dots, r). \quad (3.10)$$

Apply the second equality in Proposition 2.3 to the triple $(\nu_k, 4\epsilon, \theta)$ in place of (μ, ϵ, δ) . When $\epsilon > 0$ is small enough, we can find an integer $N_k = N_k(\nu_k, 4\epsilon, \theta) \geq 1$ such that

$$r_n(4\epsilon, \theta, \nu_k) \geq \exp(n(h_{\nu_k} - \theta)) \quad (\forall n \geq N_k).$$

This implies that if $n \geq N_k$, then the minimal number of balls $B_n(x, 4\epsilon)$ to cover $Y_k(N)$ is greater than or equal to $\exp(n(h_{\nu_k} - \theta))$. Consequently, if we use $C(n, 4\epsilon)$ to denote a maximal $(n, 4\epsilon)$ -separated set in $Y_k(N)$, then

$$\#C(n, 4\epsilon) \geq \exp(n(h_{\nu_k} - \theta)) \quad (\forall n \geq N_k). \quad (3.11)$$

Choose a sufficiently large integer N_0 such that

$$n_k := [\lambda_k n] \geq \max(N_1, \dots, N_k, N) \quad (\forall k = 1, \dots, r; \forall n \geq N_0)$$

($[\cdot]$ denoting the integral part). By the specification property, for each r points $x_1 \in C(n_1, 4\epsilon), \dots, x_r \in C(n_r, 4\epsilon)$, there exist an integer $m(\epsilon)$ depending on ϵ and a point $y = y(x_1, \dots, x_r) \in X$ such that

$$d_{n_k}(T^{a_k} y, x_k) < \epsilon \quad (1 \leq k \leq r) \quad (3.12)$$

where

$$a_1 = 0, \quad a_k = (k-1)m + \sum_{s=1}^{k-1} n_s \quad (k \geq 2).$$

Write $\hat{n} = a_r + n_r$, i.e.

$$\hat{n} = (r-1)m + \sum_{s=1}^r n_s.$$

We claim that for all such $y = y(x_1, \dots, x_r)$, we have

$$y = y(x_1, \dots, x_r) \in X_\Phi(\alpha, 5\delta, \hat{n}) \quad (3.13)$$

when n is sufficiently large, and that for two distinct points (x_1, \dots, x_r) and (x'_1, \dots, x'_r) in $C(n_1, 4\epsilon) \times \dots \times C(n_r, 4\epsilon)$, the points $y = y(x_1, \dots, x_r)$ and $y' = y(x'_1, \dots, x'_r)$ satisfy

$$d_{\hat{n}}(y, y') > 2\epsilon. \quad (3.14)$$

If we admit (3.13) and (3.14), we can conclude. In fact, the balls $B_{\hat{n}}(y, \epsilon)$ are disjoint owing to (3.14) and hence there are $\#C(n_1, 4\epsilon) \times \dots \times \#C(n_r, 4\epsilon)$ such balls. Therefore, because of (3.13), the minimal number of (\hat{n}, ϵ) -balls needed to cover $X_\Phi(\alpha, 5\delta, \hat{n})$ is greater than the number of such points y 's. That is to say

$$N(\alpha, 5\delta, \hat{n}, \epsilon) \geq \#C(n_1, 4\epsilon) \times \dots \times \#C(n_r, 4\epsilon)$$

Then by (3.11), we get

$$N(\alpha, 5\delta, \hat{n}, \epsilon) \geq \exp \sum_{k=1}^r [\lambda_k n] (h_{\nu_k} - \theta).$$

By noticing that $\frac{[\lambda_k n]}{\hat{n}} \rightarrow \lambda_k$ as $n \rightarrow \infty$ and $\sum_{k=1}^r \lambda_k = 1$, we get

$$\liminf_{\hat{n} \rightarrow \infty} \frac{1}{\hat{n}} \log N(\alpha, 5\delta, \hat{n}, \epsilon) \geq h_\mu - 3\theta$$

This, together with (3.8), implies $\Lambda_\Phi(\alpha) \geq h_\mu - 4\theta$.

Now return to prove (3.13) and (3.14). The proof of (3.14) is simple: suppose $x_k \neq x'_k$ for some $1 \leq k \leq r$. By (3.12),

$$d_{\hat{n}}(y, y') \geq d_{n_k}(T^{a_k}y, T^{a_k}y') \geq d_{n_k}(x_k, x'_k) - 2\epsilon > 4\epsilon - 2\epsilon = 2\epsilon.$$

Now prove (3.13). Recall that $\alpha_i = \int \Phi_i d\nu$ and $\nu = \sum_{k=1}^r \lambda_k \nu_k$. We have

$$|A_{\hat{n}}\Phi_i(y) - \alpha_i| \leq \left| A_{\hat{n}}\Phi_i(y) - \sum_{k=1}^r \lambda_k \int \Phi_i d\nu_k \right| + \left| \int \Phi_i d\nu - \int \Phi_i d\mu \right|.$$

Since $\tilde{d}(\mu, \nu) < \delta$ i.e. $\sum_{i=1}^\infty p_i \left| \int \Phi_i d\mu - \int \Phi_i d\nu \right| < \delta$, we have only to show that

$$\sum_{i=1}^\infty p_i \left| A_{\hat{n}}\Phi_i(y) - \sum_{k=1}^r \lambda_k \int \Phi_i d\nu_k \right| < 4\delta. \quad (3.15)$$

Write

$$\begin{aligned} A_{\hat{n}}\Phi_i(y) &= \frac{1}{\hat{n}} \sum_{k=1}^r \sum_{j=0}^{[\lambda_k n]-1} \Phi_i(T^{a_k+j}y) + \frac{1}{\hat{n}} \sum_{k=2}^r \sum_{j=a_k-m}^{a_k-1} \Phi_i(T^jy) \\ &= \sum_{k=1}^r \frac{[\lambda_k n]}{\hat{n}} A_{[\lambda_k n]}\Phi_i(T^{a_k}y) + \frac{1}{\hat{n}} \sum_{k=2}^r \sum_{j=a_k-m}^{a_k-1} \Phi_i(T^jy). \end{aligned}$$

Then

$$\left| A_{\hat{n}}\Phi_i(y) - \sum_{k=1}^r \lambda_k \int \Phi_i d\nu_k \right| \leq I_1(i) + I_2(i) + I_3(i) + I_4(i)$$

with

$$\begin{aligned}
I_1(i) &= \sum_{k=1}^r \frac{[\lambda_k n]}{\hat{n}} |A_{[\lambda_k n]} \Phi_i(T^{a_k} y) - A_{[\lambda_k n]} \Phi_i(x_k)| \\
I_2(i) &= \sum_{k=1}^r \frac{[\lambda_k n]}{\hat{n}} \left| A_{[\lambda_k n]} \Phi_i(x_k) - \int \Phi_i d\nu_k \right| \\
I_3(i) &= \sum_{k=1}^r \left| \frac{[\lambda_k n]}{\hat{n}} - \lambda_k \right| \int |\Phi_i| d\nu_k \\
I_4(i) &= \frac{1}{\hat{n}} \sum_{k=2}^r \sum_{j=a_k-m}^{a_k-1} |\Phi_i(T^j y)|.
\end{aligned}$$

Since $[\lambda_k n] \leq \lambda_k \hat{n}$ and x_k satisfies (3.12), by (3.7) we get

$$\sum_{i=1}^{\infty} p_i I_1(i) \leq \sum_{i=1}^{\infty} p_i \sum_{k=1}^r \lambda_k \text{var}(\Phi_i, \epsilon) = \sum_{i=1}^{\infty} p_i \text{var}(\Phi_i, \epsilon) < \delta.$$

Since $x_k \in Y_k(N)$ and $[\lambda_k n] \geq N$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} p_i I_2(i) &\leq \sum_{i=1}^{\infty} p_i \sum_{k=1}^r \lambda_k \left| A_{[\lambda_k n]} \Phi_i(x_k) - \int \Phi_i d\nu_k \right| \\
&= \sum_{k=1}^r \lambda_k \sum_{i=1}^{\infty} p_i \left| A_{[\lambda_k n]} \Phi_i(x_k) - \int \Phi_i d\nu_k \right| \leq \delta \sum_{k=1}^r \lambda_k = \delta.
\end{aligned}$$

Since $\|\Phi_i\| \leq 1$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} p_i I_3(i) &\leq \sum_{k=1}^r \left| \frac{[\lambda_k n]}{\hat{n}} - \lambda_k \right| < \delta \\
\sum_{i=1}^{\infty} p_i I_4(i) &\leq \frac{(r-1)m(\epsilon)}{\hat{n}} \sum_{i=1}^{\infty} p_i = \frac{(r-1)m(\epsilon)}{\hat{n}} < \delta
\end{aligned}$$

when n is sufficiently large because $\hat{n} \rightarrow \infty$ and $\frac{[\lambda_k n]}{\hat{n}} \rightarrow \lambda_k$.

By combining all these estimates, we obtain (3.15). \square

3.3. Saturatedness of systems with specification.

In this subsection we will finish our proof of Theorem 1.1, which says that systems satisfying the specification property are saturated. It remains to prove $h_{\text{top}}(G_\mu) \geq \Lambda_\Phi(\alpha)$. In fact, by Proposition 3.2, we will have $h_{\text{top}}(G_\mu) \geq h_\mu$. On the other hand, it was known to Bowen [8] that $h_\mu \geq h_{\text{top}}(G_\mu)$. So, we will get $h_{\text{top}}(G_\mu) = h_\mu$.

Proposition 3.3. $h_{\text{top}}(G_\mu) \geq \Lambda_\Phi(\alpha)$

Proof. It suffices to prove $h_{\text{top}}(G_\mu) \geq \Lambda_\Phi(\alpha) - \theta$ for any $\theta > 0$. To this end, we will construct dynamical Moran subsets of $G_\mu = X_\Phi(\alpha)$, which approach $X_\Phi(\alpha)$. The construction is based on separated sets of $X_\Phi(\alpha, \delta, n)$.

Let $m_k = m(2^{-k}\epsilon)$ be the constants in the definition of specification. By the definition of $\Lambda_\Phi(\alpha)$ (see (3.6)), when $\epsilon > 0$ is small enough there exist a sequence of positive numbers $\{\delta_k\}$ decreasing to zero and a sequence of integers $\{n_k\}$ increasing to the infinity such that

$$n_k \geq 2^{m_k} \tag{3.16}$$

and that for any $k \geq 1$ we can find a $(n_k, 5\epsilon)$ -separated set W_k of $X_\Phi(\alpha, \delta_k, n_k)$ with

$$M_k := \#W_k \geq \exp(n_k(\Lambda_\Phi(\alpha) - \theta)). \quad (3.17)$$

Choose a sequence of integers $\{N_k\}$ such that

$$N_1 = 1 \quad (3.18)$$

$$N_k \geq 2^{n_{k+1}+m_{k+1}}, \quad k \geq 2 \quad (3.19)$$

$$N_{k+1} \geq 2^{N_1 n_1 + N_2(n_2+m_2) \cdots + N_k(n_k+m_k)}, \quad k \geq 1 \quad (3.20)$$

Consider the dynamical Moran set $F = F(\epsilon, \{W_k\}, \{n_k\}, \{N_k\})$ as we constructed in the last subsection. From (3.16)-(3.20), we get

$$\liminf_{n \rightarrow \infty} \frac{1}{n} (N_1 \log M_1 + \cdots + N_k \log M_k + p \log M_{k+1}) \geq \Lambda_\Phi(\alpha) - \theta.$$

By Proposition 3.1, we have

$$h_{\text{top}}(F) \geq \Lambda_\Phi(\alpha) - \theta.$$

Thus we have only to prove $F \subseteq X_\Phi(\alpha)$. Or equivalently

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{\infty} p_i |S_n \Phi_i(x) - n\alpha_i| = 0 \quad (x \in F). \quad (3.21)$$

Let us use the same notations as in the last subsection including ℓ_k , t_k , D_k and L_k etc.

Fix $n \geq 1$. Let $k \geq 1$ and $0 \leq p < N_{k+1}$ be the integers, which depend on n , such that

$$\ell_k + p(m_{k+1} + n_{k+1}) < n \leq \ell_k + (p+1)(m_{k+1} + n_{k+1})$$

Write

$$q = n - (\ell_k + p(m_{k+1} + n_{k+1})), \quad b_s = (s-1)(m_{k+1} + n_{k+1}).$$

Decompose the interval $[0, n)$ ($\subset \mathbb{N}$) into small intervals

$$[0, n) = [0, \ell_k) \bigcup [\ell_k, \ell_k + p(m_{k+1} + n_{k+1})) \bigcup [\ell_k + p(m_{k+1} + n_{k+1}), n)$$

and decompose still $[\ell_k, \ell_k + p(m_{k+1} + n_{k+1}))$ into intervals alternatively of lengths n_{k+1} and m_{k+1} . Then cut the sum $\sum_{0 \leq j < n} \Phi_i(T^j x)$ into sums taken over small intervals. Thus we get

$$|S_n \Phi_i(x) - n\alpha_i| \leq J_1(i) + J_2(i) + J_3(i) + J_4(i)$$

where

$$\begin{aligned} J_1(i) &= |S_{\ell_k} \Phi_i(x) - \ell_k \alpha_i| \\ J_2(i) &= \sum_{s=1}^p |S_{m_{k+1}} \Phi_i(T^{\ell_k + b_s} x) - m_{k+1} \alpha_i| \\ J_3(i) &= \sum_{s=1}^p |S_{n_{k+1}} \Phi_i(T^{\ell_k + b_s + m_{k+1}} x) - n_{k+1} \alpha_i| \\ J_4(i) &= |S_q \Phi_i(T^{\ell_k + p(m_{k+1} + n_{k+1})} x) - q \alpha_i| \end{aligned}$$

Since $\|\Phi_i\| \leq 1$ (hence $|\alpha_i| \leq 1$), we have

$$J_2(i) \leq 2pm_{k+1}, \quad J_4(i) \leq 2q \leq 2(m_{k+1} + n_{k+1}).$$

By (3.19), we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{\infty} p_i J_2(i) = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{\infty} p_i J_4(i) = 0. \quad (3.22)$$

Now let us deal with $J_1(i)$ and $J_3(i)$. We claim that for any $x \in F$ there exists an $\bar{x} \in L_k$ such that

$$d_{\ell_k}(\bar{x}, x) < \frac{\epsilon}{2^{k-1}}, \quad (3.23)$$

and that for all $1 \leq s \leq p$, there exists a point $x_s \in W_{k+1}$ such that

$$d_{n_{k+1}}(x_s, T^{u_s} x) < \frac{\epsilon}{2^{k-1}} \quad (3.24)$$

where

$$u_s = \ell_k + b_s + m_{k+1}.$$

In fact, by the construction of F , there exists a point $z \in L_{k+1}$ such that

$$d_{\ell_{k+1}}(z, x) \leq \frac{\epsilon}{2^k}. \quad (3.25)$$

Assume that z descends from some $\bar{x} \in L_k$ through $y \in D_{k+1}$. Then

$$d_{\ell_k}(\bar{x}, z) < \frac{\epsilon}{2^{k+1}} \quad (3.26)$$

and

$$d_{t_{k+1}}(y, T^{\ell_k + m_{k+1}} z) < \frac{\epsilon}{2^{k+1}}. \quad (3.27)$$

On the other hand, according to the definition of D_{k+1} , there exists an $x_s \in W_{k+1}$ such that

$$d_{n_{k+1}}(x_s, T^{b_s} y) < \frac{\epsilon}{2^{k+1}}. \quad (3.28)$$

Now by the trigonometric inequality, the fact $d_{\ell_k}(z, x) \leq d_{\ell_{k+1}}(z, x)$ and (3.25) and (3.26) we get

$$d_{\ell_k}(\bar{x}, x) \leq \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^k} < \frac{\epsilon}{2^{k-1}}.$$

Thus (3.23) is proved. By (3.25), (3.27) and (3.28), we can similarly prove (3.26):

$$\begin{aligned} d_{n_{k+1}}(x_s, T^{u_s} x) &\leq d_{n_{k+1}}(x_s, T^{b_s} y) + d_{n_{k+1}}(T^{b_s} y, T^{u_s} z) + d_{n_{k+1}}(T^{u_s} z, T^{u_s} x) \\ &\leq d_{n_{k+1}}(x_s, T^{b_s} y) + d_{t_{k+1}}(y, T^{\ell_k + m_{k+1}} z) + d_{\ell_{k+1}}(z, x) \\ &< \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^k} \\ &= \frac{\epsilon}{2^{k-1}}. \end{aligned}$$

It is now easy to deal with $J_3(i)$, which is obviously bounded by

$$J_3(i) \leq \sum_{s=1}^p |S_{n_{k+1}} \Phi_i(T^{u_s} x) - S_{n_{k+1}} \Phi_i(x_s)| + \sum_{s=1}^p |S_{n_{k+1}} \Phi_i(x_s) - n_{k+1} \alpha_i|.$$

Using (3.24), we obtain

$$|S_{n_{k+1}} \Phi_i(x_s) - S_{n_{k+1}} \Phi_i(T^{u_s} x)| \leq n_{k+1} \text{var}(\Phi_i, \epsilon 2^{-(k-1)}).$$

On the other hand, since $x_s \in W_{k+1} \subseteq X_{\Phi}(\alpha, \delta_{k+1}, n_{k+1})$, we have, by definition,

$$\sum_{i=1}^{\infty} p_i |S_{n_{k+1}} \Phi_i(x_s) - n_{k+1} \alpha_i| \leq n_{k+1} \delta_{k+1}.$$

Then, combining the last three estimates, and using the facts $\sum_{j=1}^{\infty} p_j = 1$ and $pn_{k+1} \leq n$, we get

$$\frac{1}{n} \sum_{i=1}^{\infty} p_i J_3(i) \leq \sum_{i=1}^{\infty} p_i \text{var}(\Phi_i, \epsilon 2^{-(k-1)}) + \delta_{k+1}.$$

Since k can be arbitrarily large, we finally get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} p_i J_3(i) = 0. \quad (3.29)$$

Now it remains to prove

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} p_i J_1(i) = 0. \quad (3.30)$$

Observe that

$$J_1(i) \leq |S_{\ell_k} \Phi_i(x) - S_{\ell_k} \Phi_i(\bar{x})| + |S_{\ell_k} \Phi_i(\bar{x}) - \ell_k \alpha_i|.$$

By (3.23), we have $|S_{\ell_k} \Phi_i(x) - S_{\ell_k} \Phi_i(\bar{x})| \leq \ell_k \text{var}(\Phi_i, \epsilon 2^{-(k-1)})$. Then

$$J_1(i) \leq \ell_k \text{var}(\Phi_i, \epsilon 2^{-(k-1)}) + R_{k,i}$$

where

$$R_{k,i} = \max_{z \in L_k} |S_{\ell_k} \Phi_i(z) - \ell_k \alpha_i|.$$

Since $\text{var}(\Phi_i, \epsilon 2^{-(k-1)})$ tends to zero as $k \rightarrow \infty$, the desired claim (3.30) is reduced to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\infty} p_i R_{k,i} = 0. \quad (3.31)$$

We need two lemmas to estimate $R_{k,i}$.

Lemma 3.4. *For any $y \in D_{k+1}$, we have*

$$\begin{aligned} & \sum_{i=1}^{\infty} p_i |S_{t_{k+1}} \Phi_i(y) - t_{k+1} \alpha_i| \\ & \leq \sum_{i=1}^{\infty} p_i N_{k+1} n_{k+1} \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + 2(N_{k+1} - 1)m_{k+1} + N_{k+1} n_{k+1} \delta_{k+1}. \end{aligned}$$

Proof. For any $s = 1, \dots, N_{k+1}$, there exists $x_s \in W_{k+1}$ such that

$$d_{n_{k+1}}(x_s, T^{b_s} y) < \frac{\epsilon}{2^{k+1}} \quad (3.32)$$

where $b_s = (s-1)(m_{k+1} + n_{k+1})$. Write

$$S_{t_{k+1}} \Phi_i(y) = \sum_{s=1}^{N_{k+1}} S_{n_{k+1}} \Phi_i(T^{b_s} y) + \sum_{s=1}^{N_{k+1}-1} S_{m_{k+1}} \Phi_i(T^{b_s + n_{k+1}} y).$$

Then

$$\begin{aligned} & |S_{t_{k+1}} \Phi_i(y) - t_{k+1} \alpha_i| \\ & \leq \sum_{s=1}^{N_{k+1}} |S_{n_{k+1}} \Phi_i(T^{b_s} y) - n_{k+1} \alpha_i| + \sum_{s=1}^{N_{k+1}-1} |S_{m_{k+1}} \Phi_i(T^{b_s + n_{k+1}} y) - m_{k+1} \alpha_i|. \end{aligned}$$

Since $x_s \in W_{k+1} \subseteq X_\Phi(\alpha, \delta_{k+1}, n_{k+1})$, by (3.32), we have

$$\begin{aligned} & \sum_{i=1}^{\infty} p_i |S_{n_{k+1}} \Phi_i(T^{b_s} y) - n_{k+1} \alpha_i| \\ & \leq \sum_{i=1}^{\infty} p_i |S_{n_{k+1}} \Phi_i(T^{b_s} y) - S_{n_{k+1}} \Phi_i(x_s)| + \sum_{i=1}^{\infty} p_i |S_{n_{k+1}} \Phi_i(x_s) - n_{k+1} \alpha_i| \\ & \leq \sum_{i=1}^{\infty} p_i n_{k+1} \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + n_{k+1} \delta_{k+1}. \end{aligned}$$

On the other hand,

$$|S_{m_{k+1}} \Phi_i(T^{b_s + n_{k+1}} y) - m_{k+1} \alpha_i| \leq 2m_{k+1}.$$

Now it is easy to conclude. \square

Lemma 3.5.

$$\sum_{i=1}^{\infty} p_i R_{k,i} \leq 2 \sum_{i=1}^{\infty} p_i \sum_{j=1}^k \ell_j \text{var}(\Phi_i, \epsilon 2^{-j}) + 2 \sum_{j=1}^k N_j m_j + \sum_{j=1}^k \ell_j \delta_j.$$

Proof. We prove it by induction on k . When $k = 1$, we have $L_1 = D_1 = W_1 \subseteq X_\Phi(\alpha, \delta_1, n_1)$ and then

$$\sum_{i=1}^{\infty} p_i R_{1,i} \leq n_1 \delta_1 = \ell_1 \delta_1.$$

Suppose the lemma holds for k . For any $z \in L_{k+1}$ there exist $x \in L_k$ and $y \in D_{k+1}$, such that

$$d_{\ell_k}(x, z) < \frac{\epsilon}{2^{k+1}}, \quad d_{t_{k+1}}(y, T^{\ell_k + m_{k+1}} z) < \frac{\epsilon}{2^{k+1}}.$$

Write

$$S_{\ell_{k+1}} \Phi_i(z) = S_{\ell_k} \Phi_i(z) + S_{m_{k+1}} \Phi_i(T^{\ell_k} z) + S_{t_{k+1}} \Phi_i(T^{\ell_k + m_{k+1}} z).$$

Then $|S_{\ell_{k+1}} \Phi_i(z) - \ell_{k+1} \alpha_i|$ is bounded by

$$|S_{\ell_k} \Phi_i(z) - \ell_k \alpha_i| + |S_{m_{k+1}} \Phi_i(T^{\ell_k} z) - m_{k+1} \alpha_i| + |S_{t_{k+1}} \Phi_i(T^{\ell_k + m_{k+1}} z) - t_{k+1} \alpha_i|.$$

Notice that

$$\begin{aligned} |S_{\ell_k} \Phi_i(z) - \ell_k \alpha_i| & \leq |S_{\ell_k} \Phi_i(z) - S_{\ell_k} \Phi_i(x)| + |S_{\ell_k} \Phi_i(x) - \ell_k \alpha_i| \\ & \leq \ell_k \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + R_{k,i}, \end{aligned}$$

$$|S_{m_{k+1}} \Phi_i(T^{\ell_k} z) - m_{k+1} \alpha_i| \leq 2m_{k+1}$$

and

$$\begin{aligned} & |S_{t_{k+1}} \Phi_i(T^{\ell_k + m_{k+1}} z) - t_{k+1} \alpha_i| \\ & \leq |S_{t_{k+1}} \Phi_i(T^{\ell_k + m_{k+1}} z) - S_{t_{k+1}} \Phi_i(y)| + |S_{t_{k+1}} \Phi_i(y) - t_{k+1} \alpha_i| \\ & \leq t_{k+1} \text{var}(\Phi_i, \epsilon 2^{-(k+1)}) + |S_{t_{k+1}} \Phi_i(y) - t_{k+1} \alpha_i|. \end{aligned}$$

By Lemma 3.4, we have

$$\begin{aligned} \sum_{i=1}^{\infty} p_i R_{k+1,i} & \leq \sum_{i=1}^{\infty} p_i R_{k,i} + \sum_{i=1}^{\infty} p_i (\ell_k + t_{k+1} + N_{k+1} n_{k+1}) \text{var}(\Phi_i, \epsilon / 2^{k+1}) \\ & \quad + 2N_{k+1} m_{k+1} + N_{k+1} n_{k+1} \delta_{k+1}. \end{aligned}$$

Then according to the induction hypothesis the Lemma holds for $k + 1$, because

$$\ell_k + t_{k+1} \leq \ell_{k+1}, \quad N_{k+1}n_{k+1} \leq \ell_{k+1}.$$

□

Let us finish the proof of Theorem 1.1 by showing (3.31). Since $n_j \geq 2^{m_j}$, we have

$$\frac{N_j m_j}{\ell_j} \leq \frac{N_j m_j}{N_j(n_j + m_j)} = \frac{m_j}{n_j + m_j} \rightarrow 0 \quad (j \rightarrow \infty).$$

Then the estimate in Lemma 3.5 can be written as

$$\sum_{i=1}^{\infty} p_i R_{k,i} \leq \sum_{j=1}^k \ell_j c_j$$

where $c_j \rightarrow 0$ ($j \rightarrow \infty$). By (3.20), we have $\ell_k \geq 2^{\ell_{k-1}}$. It follows that

$$\frac{1}{\ell_k} \sum_{i=1}^{\infty} p_i R_{k,i} \leq c_k + \frac{1}{\ell_k} \sum_{j=1}^{k-1} c_j \ell_j.$$

This implies (3.31). □

4. VARIATIONAL PRINCIPLE

In this section, we prove variational principles for saturated systems (Theorem 1.2 and Theorem 1.3).

4.1. Proofs of Theorems 1.2 and 1.3.

The proof of Theorem 1.3 is similar to that of Theorem 1.2.

Proof of Theorem 1.2 (a). It suffices to prove that if there exists a point $x \in X_{\Phi}(\alpha; E)$, then $\mathcal{M}_{\Phi}(\alpha; E) \neq \emptyset$. That $x \in X_{\Phi}(\alpha; E)$ means

$$\limsup_{n \rightarrow \infty} \langle A_n \Phi(x), w \rangle \leq \langle \alpha, w \rangle \quad (\forall w \in E). \quad (4.1)$$

Let μ be a weak limit of $n^{-1} \sum_{j=0}^{n-1} \delta_{T^j x}$. That is to say, there exists a sequence n_m such that

$$\lim_{m \rightarrow \infty} \frac{1}{n_m} \sum_{j=0}^{n_m-1} f(T^j x) = \int f d\mu \quad (4.2)$$

for all scalar continuous functions f . We deduce from (4.1) and (4.2) that for all $w \in E$ we have

$$\int \langle \Phi, w \rangle d\mu = \lim_{m \rightarrow \infty} \langle A_{n_m} \Phi(x), w \rangle \leq \limsup_{n \rightarrow \infty} \langle A_n \Phi(x), w \rangle \leq \langle \alpha, w \rangle.$$

So $\mu \in \mathcal{M}_{\Phi}(\alpha; E)$.

Proof of Theorem 1.2 (b). Let $t = \sup_{\mu \in \mathcal{M}_{\Phi}(\alpha; E)} h_{\mu}$. What we have just proved above may be stated as follows: if $x \in X_{\Phi}(\alpha; E)$, then

$$V(x) \subset \mathcal{M}_{\Phi}(\alpha; E).$$

It follows that $h_{\mu} \leq t$ for any $\mu \in V(x)$. Thus

$$\begin{aligned} X_{\Phi}(\alpha; E) &\subset \{x \in X : \forall \mu \in V(x) \text{ satisfying } h_{\mu} \leq t\} \\ &\subset \{x \in X : \exists \mu \in V(x) \text{ satisfying } h_{\mu} \leq t\}. \end{aligned}$$

Then, due to Lemma 2.1, we get $h_{\text{top}}(X_{\Phi}(\alpha; E)) \leq t$.

Now we prove the converse inequality. For any $\mu \in \mathcal{M}_\Phi(\alpha; E)$, consider G_μ the set of generic points. We have

$$G_\mu \subset X_\Phi(\alpha; \mathbb{B}) \subset X_\Phi(\alpha; E).$$

The second inclusion is obvious and the first one is a consequence of the fact that $x \in G_\mu$ implies $\lim_{n \rightarrow \infty} A_n \Phi(x) = \int \Phi d\mu = \alpha$ in the $\sigma(\mathbb{B}^*, \mathbb{B})$ -topology. Thus

$$h_{\text{top}}(X_\Phi(\alpha; E)) \geq h_{\text{top}}(G_\mu).$$

Since μ is an arbitrary invariant measure in $\mathcal{M}_\Phi(\alpha; E)$, we can finish the proof because the system (X, T) is saturated (i.e. $h_{\text{top}}(G_\mu) = h_\mu$). \square

It is useful to point out the following facts appearing in the proof:

- (i) If $x \in X_\Phi(\alpha, E)$, then $V(x) \subset \mathcal{M}_\Phi(\alpha, E)$.
- (ii) We have

$$\bigcup_{\mu \in \mathcal{M}_\Phi(\alpha, E)} G_\mu \subset X_\Phi(\alpha, E) \subset \bigcup_{\mu \in \mathcal{M}_\Phi(\alpha, E)} \tilde{G}_\mu$$

with $\tilde{G}_\mu = \{x \in X : V(x) \ni \mu\}$. It is worth to notice the fact that all the G_μ are disjoint.

It is clear that $\mathcal{M}_\Phi(\alpha, E)$ is a compact convex subset of the space \mathcal{M}_{inv} of Borel probability invariant measures. If h_μ , as a function of μ , is upper semi-continuous on \mathcal{M}_{inv} , then the supremum in the variational principle is attained by some invariant measure, called the maximal entropy measure in $\mathcal{M}_\Phi(\alpha, E)$. Usually, the structure of $\mathcal{M}_\Phi(\alpha, E)$ is complicated. But it is sometimes possible to calculate the maximal entropy.

Proof of Theorem 1.3. Let $x \in X_\Phi^\Psi(\beta)$ and let μ be a weak limit of $n^{-1} \sum_{j=0}^{n-1} \delta_{T^j x}$. Then there exists a subsequence of integers $\{n_m\}$ such that $A_{n_m} \Phi(x)$ tends to $\int \Phi d\mu$ in the weak star topology as $m \rightarrow \infty$ because we have an expression similar to (4.2) with $f = \langle \Phi, w \rangle$ ($w \in \mathbb{B}$ being arbitrary). Hence

$$\Psi\left(\int \Phi d\mu\right) = \lim_{m \rightarrow \infty} \Psi(A_{n_m} \Phi(x)) = \lim_{n \rightarrow \infty} \Psi(A_n \Phi(x)) = \beta.$$

Thus we have proved that $\mu \in \mathcal{M}_\Phi^\Psi(\beta)$. That is to say

$$V(x) \subset \mathcal{M}_\Phi^\Psi(\beta) \quad (\forall x \in X_\Phi^\Psi(\beta)).$$

It follows that (a) holds and that due to Lemma 2.1 we have

$$h_{\text{top}}(X_\Phi^\Psi(\beta)) \leq \sup_{\mu \in \mathcal{M}_\Phi^\Psi(\beta)} h_\mu.$$

The converse inequality is a consequence of the variational principle (1.1) and the relationship

$$X_\Phi^\Psi(\beta) \supset \hat{X}_\Phi^\Psi(\beta) = \bigcup_{\alpha: \Psi(\alpha)=\beta} X_\Phi(\alpha).$$

In fact,

$$\begin{aligned} h_{\text{top}}(X_\Phi^\Psi(\beta)) &\geq h_{\text{top}}(\hat{X}_\Phi^\Psi(\beta)) \geq \sup_{\alpha: \Psi(\alpha)=\beta} h_{\text{top}}(X_\Phi(\alpha)) \\ &= \sup_{\alpha: \Psi(\alpha)=\beta} \sup_{\mu \in \mathcal{M}_\Phi(\alpha)} h_\mu \\ &= \sup_{\mu \in \mathcal{M}_\Phi^\Psi(\beta)} h_\mu. \end{aligned}$$

4.2. $\ell^\infty(\mathbb{Z})$ -valued ergodic average.

Let us consider the special case where $\mathbb{B} = \ell^1(\mathbb{Z})$. Then $\mathbb{B}^* = \ell^\infty(\mathbb{Z})$. Any $\ell^\infty(\mathbb{Z})$ -valued function Φ can be written as

$$\Phi(x) = (\Phi_n(x))_{n \in \mathbb{Z}}, \quad \text{with} \quad \sup_n |\Phi_n(x)| < \infty.$$

Recall that $\ell^\infty(\mathbb{Z})$ is equipped with the $\sigma(\ell^\infty, \ell^1)$ -topology. A $\ell^\infty(\mathbb{Z})$ -valued function Φ is continuous if and only if all coordinate functions $\Phi_n : X \rightarrow \mathbb{R}$ are continuous, because for any $w = (w_n)_{n \in \mathbb{Z}} \in \ell^1$ we have

$$\langle \Phi(x), w \rangle = \sum_{n \in \mathbb{Z}} w_n \Phi_n(x).$$

Let us give an application of the variational principle in this setting. Let I be a finite or infinite subset of positive integers. Let $\{\Phi_i\}_{i \in I}$ be a family of real continuous functions defined on X . We suppose that $\sup_{i \in I} \|\Phi_i\|_{C(X)} < \infty$. For two given sequences of real numbers $\mathbf{a} = \{a_i\}_{i \in I}$ and $\mathbf{b} = \{b_i\}_{i \in I}$, we denote by $S(\mathbf{a}, \mathbf{b})$ the set of points $x \in X$ such that

$$a_i \leq \liminf_{n \rightarrow \infty} A_n \Phi_i(x) \leq \limsup_{n \rightarrow \infty} A_n \Phi_i(x) \leq b_i \quad (\forall i \in I).$$

Corollary 4.1. *Suppose that the system (X, T) is saturated. The topological entropy of $S(\mathbf{a}, \mathbf{b})$ defined above is equal to the supremum of the measure-theoretical entropies h_μ for those invariant measures μ such that*

$$a_i \leq \int \Phi_i d\mu \leq b_i \quad (\forall i \in I).$$

Proof. For $n \in \mathbb{Z}$, let e_n be the n^{th} element of the canonical basis of $\ell^1(\mathbb{Z})$. Let Φ be a function whose n^{th} coordinate and $-n^{\text{th}}$ coordinate are respectively equal to Φ_n and $-\Phi_n$ for each $n \in I$ (other coordinates may be taken to be zero). Take the set $E \subset \ell^1$, which consists of e_i and e_{-i} for $i \in I$. Take $\alpha \in \ell^\infty$ such that $\alpha_{-i} = -a_i$ and $\alpha_i = b_i$ for $i \in I$. Now we can directly apply the variational principle by noticing that

$$\langle \Phi, e_i \rangle = \Phi_i, \quad \langle \Phi, e_{-i} \rangle = -\Phi_i \quad (i \in I).$$

□

The result contained in this corollary is new, even when I is finite. If I is finite and if $a_i = b_i$ for $i \in I$, the preceding corollary allows one to recover the results in [16] and [28].

The validity of the variational principle is to some extent equivalent to the fact that the system (X, T) is saturated.

Theorem 4.2. *Let (X, T) be a compact dynamical system. The system is saturated if and only if the variational principle (Theorem 1.2 (b)) holds for all real Banach spaces \mathbb{B} .*

Proof. It remains to prove that the variational principle implies the saturation of the system.

Take a countable set $\{f_i\}_{i \in \mathbb{N}}$ which is dense in the unit ball of $C(X)$ ($C(X)$ being the space of all real valued continuous functions on X). Consider the function

$$\Phi = (f_1, f_2, \dots, f_n, \dots)$$

which takes values in $\mathbb{B} = \ell^\infty(\mathbb{N})$. For any invariant measure $\mu \in \mathcal{M}_{\text{inv}}$, define

$$\alpha = \left(\int f_1 d\mu, \int f_2 d\mu, \dots \right) \in \ell^\infty(\mathbb{N}).$$

It is clear that $\mathcal{M}_\Phi(\alpha) = \{\mu\}$. Then the variational principle implies $h_{\text{top}}(X_\Phi(\alpha)) = h_\mu$. This finishes the proof because $X_\Phi(\alpha)$ is nothing but G_μ . \square

5. AN EXAMPLE: RECURRENCE IN AN INFINITE NUMBER OF CYLINDERS

We have got a general variational principle. In order to apply this principle, one of the main questions is to get information about the convex set $\mathcal{M}_\Phi(\alpha, E)$ and the maximal entropy measures contained in it and to compute the maximal entropy. Let us consider the symbolic dynamical system $(\{0, 1\}^\mathbb{N}, T)$, T being the shift. The structure of the space \mathcal{M}_{inv} is relatively simple. To illustrate the main result, we shall consider a special problem of recurrence into a countable number of cylinders.

5.1. Symbolic space.

Let $X = \{0, 1\}^\mathbb{N}$ and T be the shift transformation. As usual, an n -cylinder in X determined by a word $w = x_1 x_2 \dots x_n$ is denoted by $[w]$ or $[x_1, \dots, x_n]$. For any word w , define the recurrence to the cylinder $[w]$ of x by

$$R(x, [w]) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} 1_{[w]}(T^j x)$$

if the limit exists.

Let $\mathcal{W} = \{w_i\}_{i \in I}$ with $I \subset \mathbb{N}$ be a finite or infinite set of words. Let $\alpha = \{a_i\}_{i \in I}$ be a sequence of non-negative numbers. We are interested in the following recurrence set

$$R(\mathbf{a}; \mathcal{W}) = \{x \in X : R(x, [w_i]) = a_i \text{ for all } i \in I\}$$

whose topological entropy will be computed by the variational principle which takes a simpler form.

Corollary 5.1. *We have $h_{\text{top}}(R(\mathbf{a}; \mathcal{W})) = \max_{\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})} h_\mu$ where*

$$\mathcal{M}(\mathbf{a}, \mathcal{W}) = \{\mu \in \mathcal{M}_{\text{inv}} : \mu([w_i]) = a_i \text{ for all } i \in I\}.$$

Remark that the shift transformation on the symbolic space is expansive. Thus the entropy function h_μ is upper semi-continuous ([30], p. 184). Hence we can obtain the supremum in Corollary 5.1.

Recall that any Borel probability measure on X is uniquely determined by its values on cylinders. In other words, a function μ defined on all cylinders can be extended to be a Borel probability measure if and only if

$$\sum_{x_1, \dots, x_n} \mu([x_1, \dots, x_n]) = 1$$

and

$$\sum_{\epsilon \in \{0, 1\}} \mu([x_1, \dots, x_{n-1}, \epsilon]) = \mu([x_1, \dots, x_{n-1}]).$$

Such a probability measure μ is invariant if and only if

$$\sum_{\epsilon \in \{0, 1\}} \mu([\epsilon, x_2, \dots, x_n]) = \mu([x_2, \dots, x_n]).$$

The entropy h_μ of any invariant measure $\mu \in \mathcal{M}_{\text{inv}}$ can be expressed as follows

$$h_\mu = \lim_{n \rightarrow \infty} \sum_{x_1, \dots, x_n} -\mu([x_1, \dots, x_n]) \log \frac{\mu([x_1, \dots, x_n])}{\mu([x_1, \dots, x_{n-1}])}.$$

The sum in the above expression which we will denote by $h_\mu^{(n)}$ is nothing but a conditional entropy of μ and it is also the entropy of an $(n-1)$ -Markov measure μ_n , which tends towards μ as n goes to ∞ .

A Markov measure of order k is an invariant measure $\nu \in \mathcal{M}_{\text{inv}}$ having the following Markov property: for all $n > k$ and all $(x_1, \dots, x_n) \in \{0, 1\}^n$

$$\frac{\nu([x_1, \dots, x_n])}{\nu([x_1, \dots, x_{n-1}])} = \frac{\nu([x_{n-k}, \dots, x_n])}{\nu([x_{n-k}, \dots, x_{n-1}])}.$$

A Markov measure of order k is uniquely determined by its values on the $(k+1)$ -cylinders. The preceding approximating Markov measure μ_n has the same values as μ on n -cylinders.

To apply the above corollary, we have to maximize the entropy h_μ among all invariant measures μ with constraints $\mu([w_i]) = a_i$ for $i \in I$. The entropy h_μ is a function of an infinite number of variables $\mu([w])$. So we have to maximize a function of an infinite number of variables. However, in some cases it suffices to reduce the problem to maximize the conditional entropy which is a function of a finite number of variables.

Denote by $|w|$ the length of the word w . Let $\mathcal{W}_n := \{w \in \mathcal{W} : |w| \leq n\}$ and $\mathcal{M}(\mathbf{a}, \mathcal{W}_n) := \{\mu \in \mathcal{M}_{\text{inv}} : \mu([w_i]) = a_i, w_i \in \mathcal{W}_n\}$. Let μ^* be a maximal entropy measure over $\mathcal{M}(\mathbf{a}, \mathcal{W})$ and μ_n^* be the $(n-1)$ -Markov measure which converges to μ^* . Let $\mu^{(n)}$ be a maximal entropy measure over $\mathcal{M}(\mathbf{a}, \mathcal{W}_n)$. Then

$$h_{\mu^*} = \lim_{n \rightarrow \infty} h_{\mu_n^*} \leq \liminf_{n \rightarrow \infty} h_{\mu^{(n)}} \leq \limsup_{n \rightarrow \infty} h_{\mu^{(n)}} \leq h_{\mu^*}.$$

Hence

$$\lim_{n \rightarrow \infty} h_{\mu^{(n)}} = h_{\mu^*} = \max_{\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})} h_\mu.$$

However, for any measure $\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W}_n)$, we have $h_\mu = h_{\mu_n} = h_\mu^{(n)}$, where μ_n is the $(n-1)$ -Markov measure which converges to μ ([16]). Thus, $\mu^{(n)}$ is the maximal point of the conditional entropy function $h_\mu^{(n)}$.

Thus we have proved the following proposition.

Proposition 5.2. *The maximal entropy over $\mathcal{M}(\mathbf{a}, \mathcal{W})$ can be approximated by the maximal entropies over $\mathcal{M}(\mathbf{a}, \mathcal{W}_n)$'s.*

5.2. Example: Frequency of dyadic digital blocks.

Let us consider a special example:

$$\mathcal{W} = \{[0], [0^2], \dots, [0^n], \dots\}$$

where 0^k means the word with 0 repeated k times.

Theorem 5.3. *Let $\mathcal{W} = \{[0^n]\}_{n \geq 1}$ and $\mathbf{a} = \{a_n\}_{n \geq 1} \subset \mathbb{R}^+$. We have*

(a) *$R(\mathbf{a}; \{[0^n]\}_{n \geq 1}) \neq \emptyset$ if and only if*

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \quad (5.1)$$

(b) *If the above condition is satisfied, then*

$$h_{\text{top}}(R(\mathbf{a}; \mathcal{W})) = -h(1 - a_1) + \sum_{i=0}^{\infty} h(a_i - 2a_{i+1} + a_{i+2}) \quad (5.2)$$

where $h(x) = -x \log x$.

The proof of the above theorem is decomposed into several lemmas which actually allow us to find the unique invariant measure of maximal entropy and to compute its entropy.

Let μ be an invariant measure. The consistence and the invariance of the measure imply that we may partition all $(n+2)$ -cylinders into groups of the form

$$\{[0w0], [0w1], [1w0], [1w1]\}$$

such that the measures $\mu([0w0]), \mu([0w1]), \mu([1w0]), \mu([1w1])$ are linked each other through the measures $\mu([0w]), \mu([w0]), \mu([w1]), \mu([1w])$ of $(n+1)$ -cylinders. More precisely, if we write $p_w = \mu([w])$, then for any word w of length n , we have

$$\begin{aligned} p_{0w0} + p_{0w1} &= p_{0w} \\ p_{1w0} + p_{1w1} &= p_{1w} \\ p_{0w0} + p_{1w0} &= p_{w0} \\ p_{0w1} + p_{1w1} &= p_{w1} \end{aligned}$$

Lemma 5.4. *Suppose $\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})$. If $w = 0^n$ with $n \geq 0$, we have*

$$p_{00^n0} = a_{n+2} \quad (5.3)$$

$$p_{00^n1} = a_{n+1} - a_{n+2} \quad (5.4)$$

$$p_{10^n0} = a_{n+1} - a_{n+2} \quad (5.5)$$

$$p_{10^n1} = a_n - 2a_{n+1} + a_{n+2}. \quad (5.6)$$

Proof. The relation (5.4) is a consequence of the consistence

$$p_{00^n1} + p_{00^{n+1}0} = p_{00^{n+1}}$$

and the facts $p_{00^n} = a_{n+1}$ and $p_{00^{n+1}0} = a_{n+2}$; the relation (5.5) is a consequence of the invariance

$$p_{10^n0} + p_{00^{n+1}0} = p_{00^{n+1}}$$

and the same facts; to obtain the relation (5.6) we need both the invariance and the consistence:

$$p_{10^n1} + p_{00^{n+1}1} = p_{00^{n+1}} = p_{0^n} - p_{0^n0}.$$

Then by (5.4) we get

$$p_{10^n1} = p_{0^n} - p_{0^n0} - p_{00^{n+1}1} = a_n - 2a_{n+1} + a_{n+2}.$$

□

Let a, b, c be three positive numbers such that $a + b \geq c$. Consider the function

$$F(t, u, v, w) = h(t) + h(u) + h(v) + h(w) \quad (5.7)$$

defined on \mathbb{R}^4 , where $h(x) = -x \log x$.

Lemma 5.5. *Under the condition*

$$t + v = a, \quad u + w = b, \quad t + u = c$$

the function F defined by (5.7) attains its maximum at

$$t = \frac{ac}{a+b}, \quad u = \frac{bc}{a+b}, \quad v = \frac{a(a+b-c)}{a+b}, \quad w = \frac{b(a+b-c)}{a+b}. \quad (5.8)$$

Proof. From the condition we may solve t, u, v as functions of w :

$$t = c - b + w, \quad u = b - w, \quad v = a + b - c - w.$$

So, maximizing $F(t, u, v, w)$ under the condition becomes maximizing the function

$$F(w) = h(c - b + w) + h(b - w) + h(a + b - c - w) + h(w)$$

which is strictly concave in its domain. Since $h'(x) = -1 - \log x$, we have

$$F'(w) = -\log(c - b + w) + \log(b - w) + \log(a + b - c - w) - \log w.$$

Solving $F'(w) = 0$, we get $w = \frac{b(a+b-c)}{a+b}$. The corresponding t, u, v are as announced in (5.8) \square

Lemma 5.6. *Suppose that $\{a_n\}_{n \geq 0}$ is a sequence of real numbers such that*

$$1 = a_0 \geq a_1 \geq a_2 \geq \dots; \quad a_i - 2a_{i+1} + a_{i+2} \geq 0 \quad (i \geq 0). \quad (5.9)$$

There exists an invariant measure μ such that if w is not a block of 0's, we have

$$p_{\epsilon w \epsilon'} = \frac{p_{\epsilon w} p_{w \epsilon'}}{p_w} \quad (\forall \epsilon, \epsilon' \in \{0, 1\}). \quad (5.10)$$

The above recursion relations (5.10) together with (5.3-5.6) completely determine the measure μ , which is the unique maximal entropy measure among those invariant measures ν such that $\nu([0^n]) = a_n$ for $n \geq 1$.

Proof. For any $\mu \in \mathcal{M}(\mathbf{a}, \mathcal{W})$, we must have $\mu([0]) = a_1$ and $\mu([1]) = 1 - a_1$. By Proposition 5.2, we are led to find the measure $\mu^{(n+2)}$ which maximizes $h_\mu^{(n+2)}$ for each $n \geq 0$. Let μ be an arbitrary invariant measure in $\mathcal{M}(\mathbf{a}, \mathcal{W}_{n+2})$, $n \geq 0$. We identify μ with the sequence $p_w = \mu([w])$ indexed by finite words. By Lemma 5.4, we have

$$\begin{aligned} p_0 &= a_1, & p_1 &= 1 - a_1 \\ p_{00} &= a_2, & p_{01} = p_{10} &= a_1 - a_2, & p_{11} &= 1 - 2a_1 + a_2 \end{aligned}$$

So, we have

$$h_\mu^{(2)} = h(a_2) + 2h(a_1 - a_2) - h(a_1) - h(1 - a_1) + h(1 - 2a_1 + a_2). \quad (5.11)$$

Let us now consider the conditional entropy $h_\mu^{(n+2)}$ for $n \geq 1$, which is a function of μ -measures of $(n+2)$ -cylinders, whose general form is $[\epsilon w \epsilon']$ with w a word of length n and $\epsilon, \epsilon' \in \{0, 1\}$. If $w = 0^n$, by Lemma 5.4, the measures of the four cylinders $[\epsilon 0^n \epsilon']$ with $\epsilon, \epsilon' \in \{0, 1\}$ are determined by $\{a_k\}_{k \geq 1}$. If $w \neq 0^n$, the four quantities $p_{\epsilon w \epsilon'}$ are linked to each other by

$$p_{0w0} + p_{0w1} = p_{0w}, \quad p_{1w0} + p_{1w1} = p_{1w}, \quad p_{0w0} + p_{1w0} = p_{w0}$$

through measures of $(n+1)$ -cylinders: $a := p_{w0}, b := p_{1w}, c := p_{0w}$. Consider the four measures $p_{\epsilon w \epsilon'}$ as variables, there is only one free variable and three others are linked to it. Thus to any word $w \neq 0^n$ of length n is associated a free variable.

In fact, $h_\mu^{(n+2)}$ is the sum of all terms

$$-\sum_{\epsilon, \epsilon'} p_{\epsilon w \epsilon'} \log \frac{p_{\epsilon w \epsilon'}}{p_{\epsilon w}} \quad (w \in \{0, 1\}^n) \quad (5.12)$$

If $w = 0^n$, the corresponding term (5.12) is a constant depending on the sequence $\{a_n\}$ (see Lemma 5.4). If $w \neq 0^n$, there is a free variable in the term (5.12). So, maximizing $h_\mu^{(n+2)}$ is equivalent to maximizing all above terms, or equivalently to maximizing

$$-\sum_{\epsilon, \epsilon'} p_{\epsilon w \epsilon'} \log p_{\epsilon w \epsilon'}. \quad (5.13)$$

Applying Lemma 5.5 to the above function in (5.13) provides us with the maximal point $\mu^{(n+2)}$ described by (5.10) with $|w| = n$. It is easy to check that the family $\{p_w\}$ defined by (5.3-5.6) and (5.10) for all words w , verifies the consistency and the invariance conditions. The measure determined by the family is the unique measure of maximal entropy and $\mu^{(n+2)}$ is its $(n+1)$ -Markov measure. \square

Lemma 5.7. *The entropy of the invariant measure μ of maximal entropy determined in the above lemma is equal to*

$$h_\mu = -h(1 - a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2})$$

where $h(x) = -x \log x$.

Proof. Recall that for any invariant measure μ we have

$$h_\mu = \lim_{n \rightarrow \infty} h_\mu^{(n)} = \inf_n h_\mu^{(n)}$$

where

$$h_\mu^{(n)} = - \sum_{|w|=n-1} \sum_{\epsilon=0,1} p_{w\epsilon} \log \frac{p_{w\epsilon}}{p_w}.$$

Thus we may write

$$h_\mu = h_\mu^{(2)} + \sum_{n=1}^{\infty} (h_\mu^{(n+2)} - h_\mu^{(n+1)}) \quad (5.14)$$

For $n \geq 0$, write

$$\begin{aligned} h_\mu^{(n+2)} &= - \sum_{|w|=n} ' \sum_{\epsilon, \epsilon'} p_{\epsilon w \epsilon'} \log \frac{p_{\epsilon w \epsilon'}}{p_{\epsilon w}} - \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log \frac{p_{\epsilon 0^n \epsilon'}}{p_{\epsilon 0^n}} \\ &=: I_1(n) + I_2(n) \end{aligned} \quad (5.15)$$

where \sum' means that the sum is taken over $w \neq 0^n$ (0^0 meaning the empty word so that $I_1(0) = 0$). When $n = 0$, we get

$$h_\mu^{(2)} = h(a_2) + 2h(a_1 - a_2) - h(a_1) - h(1 - a_1) + h(1 - 2a_1 + a_2). \quad (5.16)$$

This coincides with (5.11).

Suppose $n \geq 1$. By the recursion relation (5.10), we have

$$\begin{aligned}
I_1(n) &= - \sum_{|w|=n} ' \sum_{\epsilon, \epsilon'} p_{\epsilon w \epsilon'} \log \frac{p_{w \epsilon'}}{p_w} \\
&= - \sum_{|w|=n} \sum_{\epsilon, \epsilon'} p_{\epsilon w \epsilon'} \log \frac{p_{w \epsilon'}}{p_w} + \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log \frac{p_{0^n \epsilon'}}{p_{0^n}} \\
&= h_\mu^{(n+1)} + I_3(n)
\end{aligned} \tag{5.17}$$

where

$$I_3(n) = \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log \frac{p_{0^n \epsilon'}}{p_{0^n}}.$$

From (5.15) and (5.17) we get

$$h_\mu^{(n+2)} - h_\mu^{(n+1)} = I_2(n) + I_3(n). \tag{5.18}$$

On the one hand, by using the invariance and the consistence we can simplify I_3 as follows

$$\begin{aligned}
I_3(n) &= \sum_{\epsilon'} p_{0^n \epsilon'} \log \frac{p_{0^n \epsilon'}}{p_{0^n}} \\
&= \sum_{\epsilon'} p_{0^n \epsilon'} \log p_{0^n \epsilon'} - p_{0^n} \log p_{0^n} \\
&= a_{n+1} \log a_{n+1} + (a_n - a_{n+1}) \log(a_n - a_{n+1}) - a_n \log a_n.
\end{aligned} \tag{5.19}$$

On the other hand, we have

$$\begin{aligned}
I_2(n) &= - \sum_{\epsilon, \epsilon'} p_{\epsilon 0^n \epsilon'} \log p_{\epsilon 0^n \epsilon'} + \sum_{\epsilon} p_{\epsilon 0^n} \log p_{\epsilon 0^n} \\
&= -a_{n+2} \log a_{n+2} - 2(a_{n+1} - a_{n+2}) \log(a_{n+1} - a_{n+2}) \\
&\quad - (a_n - 2a_{n+1} + a_{n+2}) \log(a_n - 2a_{n+1} + a_{n+2}) \\
&\quad + a_{n+1} \log a_{n+1} + (a_n - a_{n+1}) \log(a_n - a_{n+1}).
\end{aligned} \tag{5.20}$$

By combining (5.18), (5.19) and (5.20) we get

$$\begin{aligned}
\varphi(n) &:= h_\mu^{(n+2)} - h_\mu^{(n+1)} \\
&= [h(a_{n+2}) - 2h(a_{n+1}) + h(a_n)] + 2[h(a_{n+1} - a_{n+2}) - h(a_n - a_{n+1})] \\
&\quad + h(a_n - 2a_{n+1} + a_{n+2}).
\end{aligned}$$

Finally using (5.14) we get

$$h_\mu = h_\mu^{(2)} + \sum_{n=1}^{\infty} \varphi(n) = -h(1 - a_1) + \sum_{j=0}^{\infty} h(a_j - 2a_{j+1} + a_{j+2}).$$

□

Remark that the measure of maximal entropy in $R(a, \mathcal{W})$ is not necessarily ergodic. Here is an example. If $a_n = a$ ($\forall n \geq 1$) is constant, then there is a unique invariant measure in $\mathcal{M}(a, \mathcal{W})$, which is $a\delta_0 + (1-a)\delta_1$. In this case, $R(a, \mathcal{W})$ is not empty but of zero entropy. Notice that $R(a, \mathcal{W})$ contains no point in the support of the unique invariant measure. If $a = \frac{1}{2}$, $R(\frac{1}{2}, \mathcal{W})$ contains the following point

01001100011100001111...

(The terms in the two sequences $\{0^k\}$ and $\{1^k\}$ are alternatively appended.)

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